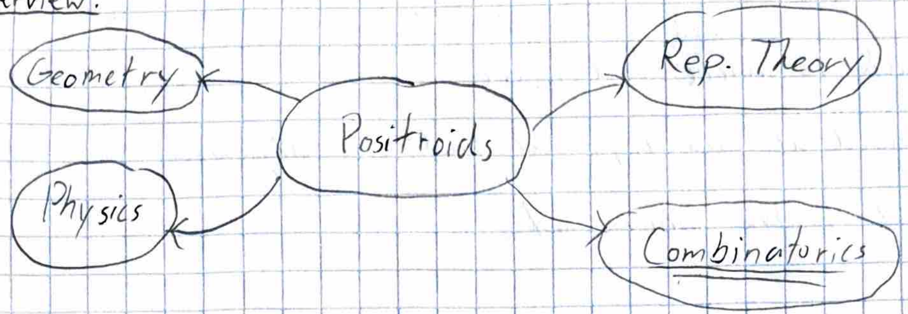
18.217 / Combinatorial Theory: Positroids / 9.9.26

General Information:

→ Professor: Alex Postnikov, apost@mit.edu

→ Website: math.mit.edu/18.217/

Overview:



Current Plan:

→ P., Total positivity, Grassmannian, and networks (2006)

→ Knutson, Lam, Speyer: Positroid varieties, juggling, and geometry, (2013)

→ P., Speyer, Williams: Matching Polytopes, Toric geometry, and the and the nonneg. part of Grassmannian (2009)

Geometry: Grassmannian, Flag Variety, Schubert cells, Richardson varieties, positroid varieties

Rep. Theory: Gelfand-Tsetlin bases, cannonical/crystal bases, tensor prod. of irreps, Demazure modules

Mostly type A

Combinatorics: Pipe dreams, SSYTs, honeycombs, positroids, plabic graphs, octahedron recurrence

Physics: Scattering amplitudes

likely mentioned at some poin in the course

→ addition (out-loud): Cluster Algebras (maybe)

→

Course Policies:

→ Problem Sets: 20% (solve a selection of problems)

→ Participation: 40% (offer notes, present PSET problems)

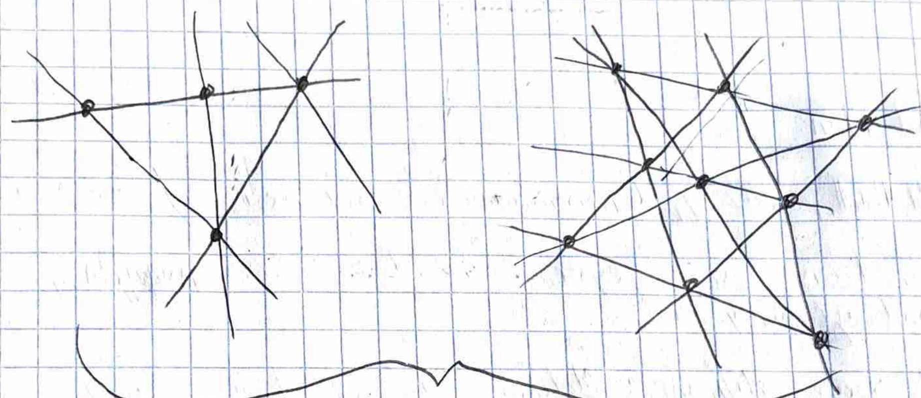
↳ are you human? Why?

→ Quizzes? 40%

Lets go back to ancient Greece...

↳ very good at geometry, but bad at algebra

↳ algebra didn't exist...

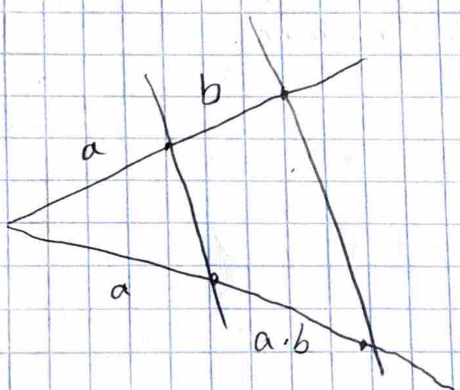


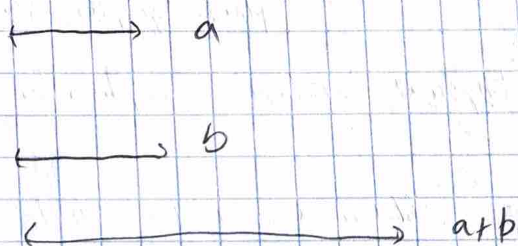
combinatorial data to extract

hyperplane configurations

Thm: (Mnev Universality Thm)

↳ the space of realizations of a matroid can be as complicated as any algebraic variety.





Can we look at some "smaller" (more managable) class of matroids and say something nice?

↳ "complicated but in a managable way" — <u>positroids</u>

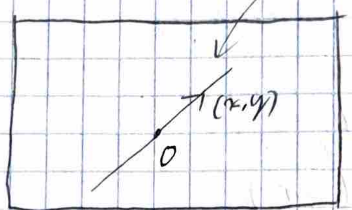
<u>Def'n (Grassmannian)</u>: ↙ let's use $\mathbb{R}$ today

↳ fix a field $(\mathbb{R}, \mathbb{C})$, and pick integers $0 \le k \le n$, Then

$Gr_{k,n}$:= the space of k-dim linear subspaces in $\mathbb{R}^n$.

<u>EX</u>: let $k=1$, $n=2$ "projective line"

$\mathbb{R}^2$



$(x:y)$

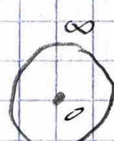
→ denoted $\mathbb{P}^1 = \{(x{:}y)\}$

$Gr_{12} = \mathbb{P}^1$

$\longrightarrow \mathbb{P}^1 = \underbrace{\{(x:1)\}}_{\mathbb{R}^1} \cup \underbrace{\{(1:0)\}}_{\infty}$

↳ this is an example of "Schubert decomposition"

Geometrically, we have an infinite point and a line



$\longrightarrow \mathbb{P}^1 = S^1$

We can also think about the Grassmannian in terms of matrices

$$Gr_{kn} = \{A : A \text{ is a } k \times n \text{ matrix of rank } k\}$$

→ <u>note</u>: two matrices A and B that can be related by "modular row operations" represent the same point.

$$A = \left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & ////// \\ 0 & & 1 & \end{array}\right]$$

$Gr_{kn} = \mathbb{R}^{k(n-k)} \cup$ lower dim things

$\dim(Gr_{kn}) = k(n-k)$

A point in Gr_{kn} can be represented by an arrangement of n hyperplanes in k-dimensional space

$$Gr_{nk} \rightsquigarrow A = \left.\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}\right\}k$$

$\to$ arrangement of n vectors $v_1, \ldots, v_n$ in $\mathbb{R}^k$

<u>Def'n (Plucker Coordinates) on Gr_{kn}</u>:

$\hookrightarrow$ Δ_I = determinant of $k \times k$ submatrix of A in columns $i_1, \ldots, i_k$

where $I = \{i_1, \ldots, i_k\} \in \binom{[n]}{k}$

Then, we can consider <u>Plucker Embeddings</u>

$$Gr_{kn} \longrightarrow \mathbb{P}^{\binom{n}{k}-1}$$

$$A \longmapsto (\Delta_I)$$

these are defined by <u>Plucker relations</u>